EEM232 Digital Systems I

Gate Circuits and Boolean Equations

Overview

- Gate Circuits and Boolean Equations
  - Binary Logic and Gates
  - Boolean Algebra
  - Standard Forms
Binary Logic and Gates

- **Binary variables** take on one of two values.
- Logical operators operate on binary values and binary variables.
- Basic logical operators are the **logic functions** AND, OR and NOT.
- Logic gates implement logic functions.
- **Boolean Algebra**: a useful mathematical system for specifying and transforming logic functions.
- We study Boolean algebra as a foundation for designing and analyzing digital systems!

Binary Variables

- Recall that the two binary values have different names:
  - True/False
  - On/Off
  - Yes/No
  - 1/0
- We use 1 and 0 to denote the two values.
- Variable identifier examples:
  - A, B, y, z, or X₁ for now
  - RESET, START_IT, or ADD1 later
Logical Operations

- The three basic logical operations are:
  - AND
  - OR
  - NOT

- AND is denoted by a dot (·).
- OR is denoted by a plus (+).
- NOT is denoted by an overbar (¯), a single quote mark (') after, or (~) before the variable.

Notation Examples

- Examples:
  - \( Y = A \cdot B \) is read “\( Y \) is equal to \( A \) AND \( B \).”
  - \( z = x + y \) is read “\( z \) is equal to \( x \) OR \( y \).”
  - \( X = \overline{A} \) is read “\( X \) is equal to NOT \( A \).”

- Note: The statement:
  \[ 1 + 1 = 2 \] (read “one plus one equals two”)
  is not the same as
  \[ 1 + 1 = 1 \] (read “one or one equals one”).
Operator Definitions

- Operations are defined on the values "0" and "1" for each operator:

**AND**

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z = X·Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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**OR**

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z = X+Y</th>
</tr>
</thead>
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</tbody>
</table>

**NOT**

<table>
<thead>
<tr>
<th>X</th>
<th>Z = X</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
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</tbody>
</table>

Truth Tables

- **Truth table** – a tabular listing of the values of a function for all possible combinations of values on its arguments

- Example: Truth tables for the basic logic operations:

**AND**

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z = X·Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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</table>

**OR**

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z = X+Y</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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</tbody>
</table>

**NOT**

<table>
<thead>
<tr>
<th>X</th>
<th>Z = X</th>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
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</tbody>
</table>
Logic Function Implementation

- Using Switches
  - For inputs:
    - logic 1 is switch closed
    - logic 0 is switch open
  - For outputs:
    - logic 1 is light on
    - logic 0 is light off.
  - NOT uses a switch such that:
    - logic 1 is switch open
    - logic 0 is switch closed

  \[ \text{Switches in parallel } => \text{ OR} \]
  \[ \text{Switches in series } => \text{ AND} \]
  \[ \text{Normally-closed switch } => \text{ NOT} \]

Logic Function Implementation (Continued)

- Example: Logic Using Switches

- Light is on \((L = 1)\) for
  \[ L(A, B, C, D) = \]
  and off \((L = 0)\), otherwise.

- Useful model for relay circuits and for CMOS gate circuits, the foundation of current digital logic technology
Logic Gates

- In the earliest computers, switches were opened and closed by magnetic fields produced by energizing coils in relays. The switches in turn opened and closed the current paths.
- Later, vacuum tubes that open and close current paths electronically replaced relays.
- Today, transistors are used as electronic switches that open and close current paths.
- Optional: Chapter 6 – Part 1: The Design Space

Logic Gate Symbols and Behavior

- Logic gates have special symbols:
  - AND gate
    - \( Z = X \cdot Y \)
  - OR gate
    - \( Z = X + Y \)
  - NOT gate or inverter
    - \( Z = \overline{X} \)

- And waveform behavior in time as follows:
  - X: \( \begin{array}{c}
    0 \ 0 \ 1 \ 1 \\
    0 \ 1 \ 0 \ 1
  \end{array} \)
  - Y: \( \begin{array}{c}
    0 \ 0 \ 0 \ 1 \\
    0 \ 1 \ 1 \ 1
  \end{array} \)
  - \( X \cdot Y \): \( \begin{array}{c}
    0 \ 0 \ 0 \ 1 \\
    0 \ 1 \ 0 \ 1
  \end{array} \)
  - \( X + Y \): \( \begin{array}{c}
    0 \ 1 \ 1 \ 1 \\
    0 \ 1 \ 1 \ 1
  \end{array} \)
  - \( \overline{X} \): \( \begin{array}{c}
    1 \ 1 \ 0 \ 0 \\
    1 \ 1 \ 0 \ 0
  \end{array} \)
Gate Delay

- In actual physical gates, if one or more input changes causes the output to change, the output change does not occur instantaneously.
- The delay between an input change(s) and the resulting output change is the gate delay denoted by $t_G$:

\[
\begin{align*}
\text{Input} & \quad 0 & \quad 1 \quad t_G \quad t_G \quad \text{Output} & \quad 0 & \quad 1 \\
0 & & 0 & & 0 & & 0 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{align*}
\]

\[t_G = 0.3 \text{ ns}\]

Logic Diagrams and Expressions

- Boolean equations, truth tables and logic diagrams describe the same function!
- Truth tables are unique; expressions and logic diagrams are not. This gives flexibility in implementing functions.

\[
F = X + \overline{Y} \cdot Z
\]

Truth Table

\[
\begin{array}{ccc|c}
X & Y & Z & F \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

Logic Diagram
Boolean Algebra

- An algebraic structure defined on a set of at least two elements, B, together with three binary operators (denoted +, · and ) that satisfies the following basic identities:

| 1. | $X + 0 = X$ | 2. | $X \cdot 1 = X$ |
| 3. | $X + 1 = 1$ | 4. | $X \cdot 0 = 0$ |
| 5. | $X + X = X$ | 6. | $X \cdot X = X$ |
| 7. | $X + \overline{X} = 1$ | 8. | $X \cdot \overline{X} = 0$ |
| 9. | $\overline{\overline{X}} = X$ |
| 10. | $X + Y = Y + X$ | 11. | $XY = YX$ | Commutative |
| 12. | $(X + Y) + Z = X + (Y + Z)$ | 13. | $(XY)Z = X(YZ)$ | Associative |
| 14. | $X(Y + Z) = XY + XZ$ | 15. | $X + YZ = (X + Y)(X + Z)$ | Distributive |
| 16. | $\overline{X + Y} = \overline{X} \cdot \overline{Y}$ | 17. | $\overline{X + Y} = \overline{X} + \overline{Y}$ | DeMorgan’s |

Boolean Operator Precedence

- The order of evaluation in a Boolean expression is:
  1. Parentheses
  2. NOT
  3. AND
  4. OR

- Consequence: Parentheses appear around OR expressions

- Example: $F = A(B + C)(C + \overline{D})$
Useful Theorems

- \( x \cdot y + \overline{x} \cdot y = y \) (Minimization)
- \( x + x \cdot y = x \) (Absorption)
- \( x + \overline{x} \cdot y = x + y \) (Simplification)
- \( x \cdot y + \overline{x} \cdot z + y \cdot z = x \cdot y + \overline{x} \cdot z \) (Consensus)
- \( (x + y) \cdot (\overline{x} + z) \cdot (y + z) = (x + y) \cdot (\overline{x} + z) \)
- \( \overline{x + y} = \overline{x} \cdot \overline{y} \) (DeMorgan's Laws)

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Boolean Function Evaluation

\[
\begin{array}{cccccc}
F1 &=& xy\overline{z} \\
F2 &=& x + \overline{y}z \\
F3 &=& \overline{x}y\overline{z} + \overline{x}y z + xy \\
F4 &=& xy\overline{y} + \overline{x}z \\
\end{array}
\]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>F1</th>
<th>F2</th>
<th>F3</th>
<th>F4</th>
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<tbody>
<tr>
<td>0</td>
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Expression Simplification

- An application of Boolean algebra
- Simplify to contain the smallest number of literals (complemented and uncomplemented variables):

\[ AB \cdot \overline{A} \cdot C \cdot D + \overline{A} \cdot B \cdot D + \overline{A} \cdot C \cdot \overline{D} + A \cdot B \cdot C \cdot D \]
\[ = AB + ABCD + \overline{A} \cdot C \cdot D + \overline{A} \cdot C \cdot \overline{D} + A \cdot B \cdot D \]
\[ = AB + AB(\overline{C} \cdot D) + \overline{A} \cdot C \cdot (D + \overline{D}) + \overline{A} \cdot B \cdot D \]
\[ = AB + \overline{A} \cdot C + \overline{A} \cdot B \cdot D = B(A + \overline{A} \cdot D) + \overline{A} \cdot C \]
\[ = B(A + D) + \overline{A} \cdot C \quad 5 \text{ literals} \]

Complementing Functions

- Use DeMorgan's Theorem to complement a function:
  1. Interchange AND and OR operators
  2. Complement each constant value and literal

- Example: Complement \( F = (x + y + z)(x + y + z) \)

\[ \overline{F} = (\overline{x} + \overline{y} + \overline{z})(\overline{x} + \overline{y} + \overline{z}) \]

- Example: Complement \( G = (\overline{a} + bc)\overline{d} + e \)

\[ \overline{G} = \]
Overview – Canonical Forms

- What are Canonical Forms?
- Minterms and Maxterms
- Index Representation of Minterms and Maxterms
- Sum-of-Minterm (SOM) Representations
- Product-of-Maxterm (POM) Representations
- Representation of Complements of Functions
- Conversions between Representations

Canonical Forms

- It is useful to specify Boolean functions in a form that:
  - Allows comparison for equality.
  - Has a correspondence to the truth tables
- Canonical Forms in common usage:
  - Sum of Minterms (SOM)
  - Product of Maxterms (POM)
Minterms

- **Minterms** are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., \( x \)) or complemented (e.g., \( \overline{x} \)), there are \( 2^n \) minterms for \( n \) variables.
- **Example:** Two variables (X and Y) produce \( 2 \times 2 = 4 \) combinations:
  - \( XY \) (both normal)
  - \( X \overline{Y} \) (X normal, Y complemented)
  - \( \overline{X} Y \) (X complemented, Y normal)
  - \( \overline{X} \overline{Y} \) (both complemented)
- Thus there are four minterms of two variables.

Maxterms

- **Maxterms** are OR terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., \( x \)) or complemented (e.g., \( \overline{x} \)), there are \( 2^n \) maxterms for \( n \) variables.
- **Example:** Two variables (X and Y) produce \( 2 \times 2 = 4 \) combinations:
  - \( X + Y \) (both normal)
  - \( X + \overline{Y} \) (x normal, y complemented)
  - \( \overline{X} + Y \) (x complemented, y normal)
  - \( \overline{X} + \overline{Y} \) (both complemented)
Maxterms and Minterms

- Examples: Two variable minterms and maxterms.

<table>
<thead>
<tr>
<th>Index</th>
<th>Minterm</th>
<th>Maxterm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\overline{x} \overline{y}$</td>
<td>$x + y$</td>
</tr>
<tr>
<td>1</td>
<td>$\overline{x} y$</td>
<td>$x + \overline{y}$</td>
</tr>
<tr>
<td>2</td>
<td>$x \overline{y}$</td>
<td>$\overline{x} + y$</td>
</tr>
<tr>
<td>3</td>
<td>$x y$</td>
<td>$\overline{x} + \overline{y}$</td>
</tr>
</tbody>
</table>

- The index above is important for describing which variables in the terms are true and which are complemented.

Standard Order

- Minterms and maxterms are designated with a subscript
- The subscript is a number, corresponding to a binary pattern
- The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.
- All variables will be present in a minterm or maxterm and will be listed in the same order (usually alphabetically)
- Example: For variables a, b, c:
  - Maxterms: $(a + b + \overline{c})$, $(a + b + c)$
  - Terms: $(b + a + c)$, $a \overline{c} b$, and $(c + b + a)$ are NOT in standard order.
  - Minterms: $a \overline{b} c$, $ab c$, $\overline{a} \overline{b} c$
  - Terms: $(a + c)$, $\overline{b} c$, and $(\overline{a} + b)$ do not contain all variables
Purpose of the Index

- The index for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.
- For Minterms:
  - “1” means the variable is “Not Complemented” and
  - “0” means the variable is “Complemented”.
- For Maxterms:
  - “0” means the variable is “Not Complemented” and
  - “1” means the variable is “Complemented”.

Index Example in Three Variables

- Example: (for three variables)
- Assume the variables are called X, Y, and Z.
- The standard order is X, then Y, then Z.
- The Index 0 (base 10) = 000 (base 2) for three variables. All three variables are complemented for minterm 0 ( ̅X, ̅Y, ̅Z) and no variables are complemented for maxterm 0 (X,Y,Z).
  - Minterm 0, called m₀ is XYZ.
  - Maxterm 0, called M₀ is (X + Y + Z).
  - Minterm 6?
  - Maxterm 6?
Index Examples – Four Variables

<table>
<thead>
<tr>
<th>Index</th>
<th>Binary Pattern</th>
<th>Minterm $m_i$</th>
<th>Maxterm $M_i$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0000</td>
<td>$\overline{a}{\overline{b}{\overline{c}}{d}}$</td>
<td>$a + b + c + d$</td>
</tr>
<tr>
<td>1</td>
<td>0001</td>
<td>$\overline{a}{b}{c}{d}$</td>
<td>$?$</td>
</tr>
<tr>
<td>3</td>
<td>0011</td>
<td>$?$</td>
<td>$a + b + \overline{c} + \overline{d}$</td>
</tr>
<tr>
<td>5</td>
<td>0101</td>
<td>$a{b}{c}{d}$</td>
<td>$a + \overline{b} + c + \overline{d}$</td>
</tr>
<tr>
<td>7</td>
<td>0111</td>
<td>$?$</td>
<td>$a + b + \overline{c} + \overline{d}$</td>
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<tr>
<td>10</td>
<td>1010</td>
<td>$\overline{a}{b}{c}{d}$</td>
<td>$a + b + \overline{c} + \overline{d}$</td>
</tr>
<tr>
<td>13</td>
<td>1101</td>
<td>$a{b}\overline{c}{d}$</td>
<td>$?$</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
<td>$a{b}{c}{d}$</td>
<td>$a + \overline{b} + \overline{c} + \overline{d}$</td>
</tr>
</tbody>
</table>

Minterm and Maxterm Relationship

- **Review: DeMorgan's Theorem**
  \[ \overline{x \cdot y} = \overline{x} + \overline{y} \text{ and } \overline{x + y} = \overline{x} \cdot \overline{y} \]

- **Two-variable example:**
  \[ M_2 = \overline{x} + y \text{ and } m_2 = x \cdot \overline{y} \]
  Thus $M_2$ is the complement of $m_2$ and vice-versa.

- **Since DeMorgan's Theorem holds for $n$ variables, the above holds for terms of $n$ variables**

- **giving:**
  \[ M_i = \overline{m}_i \text{ and } m_i = \overline{M}_i \]
  Thus $M_i$ is the complement of $m_i$. 
Function Tables for Both

- Minterms of 2 variables

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>m₀</th>
<th>m₁</th>
<th>m₂</th>
<th>m₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
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<td>1</td>
</tr>
</tbody>
</table>

- Maxterms of 2 variables

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>M₀</th>
<th>M₁</th>
<th>M₂</th>
<th>M₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- Each column in the maxterm function table is the complement of the column in the minterm function table since Mi is the complement of mi.

Observations

- In the function tables:
  - Each minterm has one and only one 1 present in the \(2^n\) terms (a minimum of 1s). All other entries are 0.
  - Each maxterm has one and only one 0 present in the \(2^n\) terms. All other entries are 1 (a maximum of 1s).

- We can implement any function by "ORing" the minterms corresponding to "1" entries in the function table. These are called the minterms of the function.

- We can implement any function by "ANDing" the maxterms corresponding to "0" entries in the function table. These are called the maxterms of the function.

- This gives us two canonical forms:
  - Sum of Minterms (SOM)
  - Product of Maxterms (POM)

for stating any Boolean function.
Minterm Function Example

- Example: Find $F_1 = m_1 + m_4 + m_7$
- $F_1 = \overline{x} \overline{y} z + x \overline{y} \overline{z} + x y z$

<table>
<thead>
<tr>
<th>$x$ $y$ $z$</th>
<th>index</th>
<th>$m_1$ $m_4$ $m_7$</th>
<th>$F_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0</td>
<td>0</td>
<td>0 + 0 + 0</td>
<td>0</td>
</tr>
<tr>
<td>0 0 1</td>
<td>1</td>
<td>1 + 0 + 0</td>
<td>1</td>
</tr>
<tr>
<td>0 1 0</td>
<td>2</td>
<td>0 + 0 + 0</td>
<td>0</td>
</tr>
<tr>
<td>0 1 1</td>
<td>3</td>
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<td>0</td>
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<tr>
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<td>1</td>
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<tr>
<td>1 0 1</td>
<td>5</td>
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<td>0</td>
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<tr>
<td>1 1 0</td>
<td>6</td>
<td>0 + 0 + 0</td>
<td>0</td>
</tr>
<tr>
<td>1 1 1</td>
<td>7</td>
<td>0 + 0 + 1</td>
<td>1</td>
</tr>
</tbody>
</table>

Minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$
- $F(A, B, C, D, E) =$
Maxterm Function Example

- Example: Implement \( F_1 \) in maxterms:
  \[
  F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6
  \]
  \[
  F_1 = (x + y + z) \cdot (x + \overline{y} + z) \cdot (x + \overline{y} + \overline{z})
  \]
  \[
  \cdot (\overline{x} + y + \overline{z}) \cdot (\overline{x} + \overline{y} + z)
  \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( i )</th>
<th>( M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6 = F_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0.1.1.1.1 = 0</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>1.1.1.1.1 = 1</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>2</td>
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</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1.1.0.1.1 = 0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>4</td>
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</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>1.1.1.1.0 = 0</td>
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<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>1.1.1.1.0 = 0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>1.1.1.1.1 = 1</td>
</tr>
</tbody>
</table>

Maxterm Function Example

- \( F(A, B, C, D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14} \)
- \( F(A, B, C, D) = \)
**Canonical Sum of Minterms**

- Any Boolean function can be expressed as a **Sum of Minterms**.
  - For the function table, the minterms used are the terms corresponding to the 1's
  - For expressions, **expand** all terms first to explicitly list all minterms. Do this by “ANDing” any term missing a variable v with a term \((v + \overline{v})\).

- **Example**: Implement \( f = x + \overline{x} \overline{y} \) as a sum of minterms.
  - First expand terms: \( f = x(y + \overline{y}) + \overline{x} \overline{y} \)
  - Then distribute terms: \( f = xy + x\overline{y} + \overline{x} \overline{y} \)
  - Express as sum of minterms: \( f = m_3 + m_2 + m_0 \)

---

**Another SOM Example**

- **Example**: \( F = A + \overline{B} \overline{C} \)
  - There are three variables, A, B, and C which we take to be the standard order.
  - Expanding the terms with missing variables:

  - Collect terms (removing all but one of duplicate terms):
  - Express as SOM:
Shorthand SOM Form

- From the previous example, we started with:
  \[ F = A + \overline{B} C \]
- We ended up with:
  \[ F = m_1+m_4+m_5+m_6+m_7 \]
- This can be denoted in the formal shorthand:
  \[ F(A, B, C) = \Sigma_m(1,4,5,6,7) \]
- Note that we explicitly show the standard variables in order and drop the “m” designators.

Canonical Product of Maxterms

- Any Boolean Function can be expressed as a Product of Maxterms (POM).
  - For the function table, the maxterms used are the terms corresponding to the 0's.
  - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law, “ORing” terms missing variable \( v \) with a term equal to \( V \cdot V \) and then applying the distributive law again.
- Example: Convert to product of maxterms:
  \[ f(x, y, z) = x + \overline{x} \overline{y} \]
  Apply the distributive law:
  \[ x + \overline{x} \overline{y} = (x + \overline{x})(x + \overline{y}) = 1 \cdot (x + \overline{y}) = x + \overline{y} \]
  Add missing variable \( z \):
  \[ x + \overline{y} + z \cdot \overline{z} = (x + \overline{y} + z)(x + \overline{y} + \overline{z}) \]
  Express as POM: \( f = M_2 \cdot M_3 \)
Another POM Example

- Convert to Product of Maxterms:
  
  \[ f(A, B, C) = A \overline{C} + BC + \overline{A} \overline{B} \]

- Use \[ x + y z = (x+y) \cdot (x+z) \] with \( x = (A \overline{C} + BC), \ y = \overline{A}, \) and \( z = \overline{B} \) to get:
  
  \[ f = (A \overline{C} + BC + \overline{A})(A \overline{C} + BC + \overline{B}) \]

- Then use \( x + \overline{x} y = x + y \) to get:
  
  \[ f = (\overline{C} + BC + \overline{A})(A \overline{C} + C + \overline{B}) \]

  and a second time to get:
  
  \[ f = (\overline{C} + B + \overline{A})(A + C + \overline{B}) \]

- Rearrange to standard order,
  
  \[ f = (\overline{A} + B + \overline{C})(A + \overline{B} + C) \] to give \( f = M_5 \cdot M_2 \)

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.

- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.

- Example: Given \( F(x, y, z) = \Sigma_m(1,3,5,7) \)
  
  \[ \overline{F}(x, y, z) = \Sigma_m(0,2,4,6) \]
  
  \[ \overline{F}(x, y, z) = \Pi_m(1,3,5,7) \]
Conversion Between Forms

- To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
  - Find the function complement by swapping terms in the list with terms not in the list.
  - Change from products to sums, or vice versa.

- Example: Given \( F \) as before:
  - Form the Complement: \( F(x, y, z) = \Sigma_m(1,3,5,7) \)
  - Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: \( F(x, y, z) = \Pi_m(0,2,4,6) \)

Standard Forms

- **Standard Sum-of-Products (SOP) form:** equations are written as an OR of AND terms
- **Standard Product-of-Sums (POS) form:** equations are written as an AND of OR terms

- **Examples:**
  - SOP: \( A B C + \overline{A} \overline{B} C + B \)
  - POS: \( (A+B) \cdot (A+B+C) \cdot C \)

- These “mixed” forms are neither SOP nor POS
  - \( (A B + C) (A + C) \)
  - \( A B C + A C (A + B) \)
Standard Sum-of-Products (SOP)

- A sum of minterms form for \( n \) variables can be written down directly from a truth table.
  - Implementation of this form is a two-level network of gates such that:
    - The first level consists of \( n \)-input AND gates, and
    - The second level is a single OR gate (with fewer than \( 2^n \) inputs).
- This form often can be simplified so that the corresponding circuit is simpler.

A Simplification Example:

- \( F(A, B, C) = \Sigma m(1, 4, 5, 6, 7) \)
- Writing the minterm expression:
  \[ F = \overline{A} \overline{B} C + A \overline{B} \overline{C} + A \overline{B} C + \overline{A} B \overline{C} + ABC \]
- Simplifying:
  \[ F = \]

- Simplified \( F \) contains 3 literals compared to 15 in minterm \( F \)
**AND/OR Two-level Implementation of SOP Expression**

- The two implementations for $F$ are shown below – it is quite apparent which is simpler!

**SOP and POS Observations**

- The previous examples show that:
  - Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity
  - Boolean algebra can be used to manipulate equations into simpler forms.
  - Simpler equations lead to simpler two-level implementations

- Questions:
  - How can we attain a “simplest” expression?
  - Is there only one minimum cost circuit?
  - The next part will deal with these issues.
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